STEP MATHEMATICS 3

2023

Mark Scheme

1. (i) The line through P and Q is

$$\frac{y-ap^2}{x-2ap}=\frac{y-aq^2}{x-2aq}$$

Alternatives

$$\frac{y-ap^2}{x-2ap}=\frac{ap^2-aq^2}{2ap-2aq}=\frac{p+q}{2}$$

$$\frac{y-aq^2}{x-2aq}=\frac{ap^2-aq^2}{2ap-2aq}=\frac{p+q}{2}$$

or multiplied to remove denominators.

M1

$$(y-ap^2)(x-2aq)=(y-aq^2)(x-2ap)$$

$$(2ap-2aq)y+2a^2(p^2q-pq^2)=(ap^2-aq^2)x$$

P and Q are distinct thus $p\neq q$ and so $p-q\neq 0$

Therefore $2y+2apq=(p+q)x$ that is $(p+q)x-2y-2apq=0$

A1

The perpendicular distance of $(0,3a)$ from the line PQ is $2a$ which requires

$$\left|\frac{-6a-2apq}{\sqrt{((p+q)^2+4)}}\right|=2a$$

M1 A1 A1 A1

that is $(pq+3)^2=(p+q)^2+4$ **M1 A1**

i.e. $(p+q)^2=p^2q^2+6pq+9-4=p^2q^2+6pq+5$ (*)

A1* (9)

Alternatives M1A1 as before

(I) $(p+q)x-2y-2apq=0$ meets $x^2+(y-3a)^2=4a^2$ when

$$4(y+apq)^2+(p+q)^2(y^2-6ay+5a^2)=0$$

M1 A1

$$(4+(p+q)^2)y^2-(6a(p+q)^2-8apq)y+(5a^2(p+q)^2+4a^2p^2q^2)=0$$

A1

Thus using $(p+q)^2=p^2q^2+6pq+5$, **M1**

$$(pq+3)^2y^2-2a(pq+3)(3pq+5)y+a^2(3pq+5)^2=0$$

which is a perfect square, **A1**

so $(pq+3)y - a(3pq+5) = 0$ which only has a single root so the line is a tangent. **A1 A1***

(II) Foot of perpendicular from $(0,3a)$ to $(p+q)x - 2y - 2apq = 0$ is at intersection with $(p+q)y + 2x = 3a(p+q)$ **M1 A1**

So solving $(p+q)^2y + 4y + 4apq = 3a(p+q)^2$ **A1**

$y = \frac{3a(p+q)^2 - 4apq}{(p+q)^2+4}$ and $x = \frac{(p+q)}{2}\left(3a - \frac{3a(p+q)^2-4apq}{(p+q)^2+4}\right) = \frac{2a(p+q)(3+pq)}{(p+q)^2+4}$

A1

and so the square of the distance is **M1 A1**

$$\left[\frac{2a(p+q)(3+pq)}{(p+q)^2+4}\right]^2 + \left[\frac{4a(3+pq)}{(p+q)^2+4}\right]^2 = \left[\frac{2a(3+pq)}{(p+q)^2+4}\right]^2 (4+(p+q)^2)$$
$$= \frac{(3+4a^2pq)^2}{(4+(p+q)^2)} = 4a^2$$

using given condition. **A1***

(III) Method is possible by differentiation of circle equation. Partial or incorrect solution by this method zero marks; completely correct solution full marks; completely correct solution except minor inaccuracy, withhold one accuracy mark and final accuracy mark.

(ii) (*) can be re-written

$$q^2(p^2-1) + 4pq + (5-p^2) = 0$$

M1

Considering this as a quadratic equation for q, to be two distinct roots, $p^2 - 1 \neq 0$ (it is given that $p^2 \neq 1$) **E1** and the discriminant needs to be positive.

$$16p^2 - 4(p^2-1)(5-p^2) = 4(p^4 - 2p^2 + 5) = 4(p^2-1)^2 + 16 > 0$$

as required. **M1 A1**

$q_1 + q_2 = \frac{-4p}{(p^2-1)}$, $q_1q_2 = \frac{(5-p^2)}{(p^2-1)}$ **A1 A1 (6)**

(iii) Given P, with $p^2 \neq 1$, by (ii) points Q_1 and Q_2 can be defined with parameters q_1 and q_2 where q_1 and q_2 are the roots of (*). So by (i), PQ_1 and PQ_2 are tangents to the circle centre $(0,3a)$ radius $2a$. **E1**

The perpendicular distance of $(0,3a)$ from the line Q_1Q_2 is

$$\left|\frac{-6a - 2aq_1q_2}{\sqrt{((q_1+q_2)^2+4)}}\right| = \left|\frac{-6a - 2a\frac{(5-p^2)}{(p^2-1)}}{\sqrt{\left(\left(\frac{-4p}{(p^2-1)}\right)^2+4\right)}}\right| = 2a\left|\frac{3(p^2-1)+(5-p^2)}{\sqrt{16p^2+4(p^2-1)^2}}\right|$$

M1 A1

$$= 2a\left|\frac{2p^2+2}{\sqrt{4p^4+16p^2+4}}\right| = 2a$$

A1

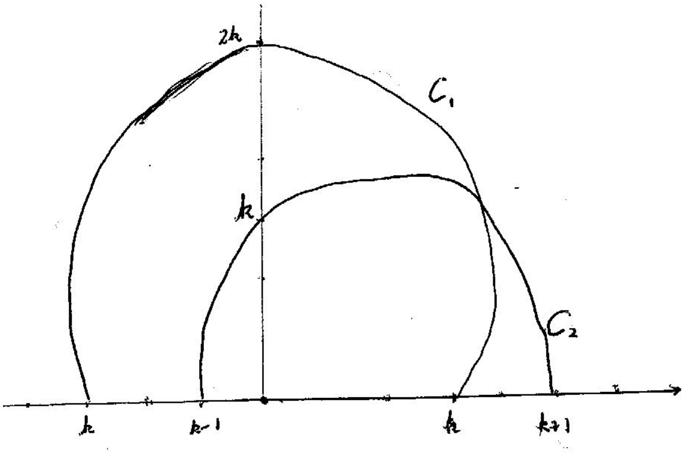
Alternative Q_1Q_2 is the third such line provided that $(q_1q_2+3)^2 = (q_1+q_2)^2+4$

$$(q_1q_2+3)^2-(q_1+q_2)^2-4 = \left[\frac{(5-p^2)}{(p^2-1)}+3\right]^2 - \left[\frac{-4p}{(p^2-1)}\right]^2 - 4$$
$$= \frac{4(p^2+1)^2-16p^2-4(p^2-1)^2}{(p^2-1)^2} = 0$$

M1 A1 A1

Thus PQ_1Q_2 is the triangle required. **E1 (5)**

2. (i)



G1 G1 G1 G1

At intersection, when $\theta = \alpha$, $k(1 + \sin\theta) = k + \cos\theta$

Therefore, $k\sin\alpha = \cos\alpha$, that is, $\tan\alpha = \frac{1}{k}$ **B1* (5)**

(ii) Area A is

$$\frac{1}{2}\int_0^\alpha \left(k(1+\sin\theta)\right)^2 d\theta = \frac{k^2}{2}\int_0^\alpha 1 + 2\sin\theta + \sin^2\theta \, d\theta$$

M1

$$= \frac{k^2}{2}\int_0^\alpha 1 + 2\sin\theta + \frac{1-\cos 2\theta}{2} \, d\theta = \frac{k^2}{2}\left[\frac{3}{2}\theta - 2\cos\theta - \frac{1}{4}\sin 2\theta\right]_0^\alpha$$

dM1 **A1**

$$= \frac{k^2}{2}\left\{\frac{3}{2}\alpha - 2\cos\alpha - \frac{1}{4}\sin 2\alpha + 2\right\} = \frac{k^2}{2}\left\{\frac{3}{2}\alpha - 2\cos\alpha - \frac{1}{2}\sin\alpha\cos\alpha + 2\right\}$$

$$= \frac{k^2}{4}(3\alpha - \sin\alpha\cos\alpha) + k^2(1-\cos\alpha)$$

A1* (4)

(iii) Area B is

$$\frac{1}{2}\int_\alpha^\pi (k+\cos\theta)^2 d\theta = \frac{1}{2}\int_\alpha^\pi k^2 + 2k\cos\theta + \cos^2\theta \, d\theta = \frac{1}{2}\int_\alpha^\pi k^2 + 2k\cos\theta + \frac{1+\cos 2\theta}{2} \, d\theta$$

M1 **M1**

$$= \frac{1}{2}\left[k^2\theta + 2k\sin\theta + \frac{1}{2}\theta + \frac{1}{4}\sin 2\theta\right]_\alpha^\pi = \frac{1}{2}\left\{k^2\pi + \frac{\pi}{2} - k^2\alpha - 2k\sin\alpha - \frac{\alpha}{2} - \frac{1}{4}\sin 2\alpha\right\}$$

A1

$$= \frac{1}{2}\left\{k^2\pi + \frac{\pi}{2} - k^2\alpha - 2k\sin\alpha - \frac{\alpha}{2} - \frac{1}{2}\sin\alpha\cos\alpha\right\}$$

$$= \frac{1}{4}\{2k^2\pi + \pi - 2k^2\alpha - 4k\sin\alpha - \alpha - \sin\alpha\cos\alpha\}$$

A1 (4)

(iv) As $k \to \infty$, α is small as $\tan\alpha = \frac{1}{k}$ so $\alpha \approx \sin\alpha \approx \tan\alpha = \frac{1}{k}$ and $\cos\alpha \approx 1 - \frac{1}{2k^2}$ **M1**

Area A is $\frac{k}{2}$ + terms of lower order in k **A1**

Area B is $\frac{k^2\pi}{2}$ + terms of lower order in k **A1**

So, area R is $\frac{k^2\pi}{2}$ + terms of lower order in k

Area T is

$$\frac{1}{2}\int_0^{\pi}(k+\cos\theta)^2 d\theta = \frac{1}{4}(2k^2\pi + \pi)$$

or alternatively, use of result from (iii) with $\alpha = 0$

which is $\frac{k^2\pi}{2}$ + terms of lower order in k **B1**

Thus, as required,

$$\frac{area\ of\ R}{area\ of\ T} = \frac{\frac{k^2\pi}{2} + \text{ terms of lower order in } k}{\frac{k^2\pi}{2} + \text{ terms of lower order in } k} \to 1$$

E1

Area S is

$$\frac{1}{2}\int_0^{\pi}\big(k(1+\sin\theta)\big)^2 d\theta = \frac{k^2}{4} \times 3\pi + 2k^2 = k^2\left(\frac{3\pi}{4} + 2\right)$$

or alternatively, use of result from (ii) with $\alpha = \pi$

B1

Thus

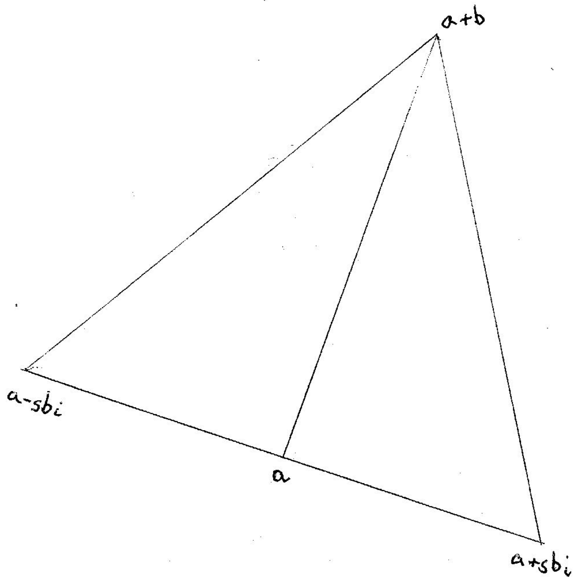
$$\frac{area\ of\ R}{area\ of\ S} \to \frac{\frac{\pi}{2}}{\left(\frac{3\pi}{4} + 2\right)} = \frac{2\pi}{3\pi + 8}$$

A1 (7)

3. (i) $sb\mathrm{i}$ represents a vector perpendicular to the vector represented by b. **E1** Thus, the two points represented by $a \pm sb\mathrm{i}$ are equidistant from the point represented by a **E1** and they are joined to it by vectors which are perpendicular to that joining it to C so they form a base of a triangle which has altitude from a to $a + b$ and has two equal length sides, by Pythagoras. **E1 (3)**

Alternative Distance $a + b$ to $a + sb\mathrm{i}$ is $|(a + sb\mathrm{i}) - (a + b)| = |b||s\mathrm{i} - 1| = |b|\sqrt{s^2 + 1}$ as s is real, **E1** and distance $a + b$ to $a - sb\mathrm{i}$ is $|(a - sb\mathrm{i}) - (a + b)| = |b||-s\mathrm{i} - 1| = |b|\sqrt{s^2 + 1}$, **E1** so two equal length sides. **E1**

a is represented by the midpoint of the base. **B1** b is represented by the vector joining the midpoint of the base to the other vertex. **B1** s is the scale factor that the magnitude of the altitude is multiplied by to obtain half the base. **B1**



(3)

(ii) We require complex a and b and real s such that

$$(a + sb\mathrm{i}) + (a - sb\mathrm{i}) + (a + b) = 0 \ \Rightarrow b = -3a$$

M1 **A1**

and

$$(a + sb\mathrm{i})(a - sb\mathrm{i}) + (a - sb\mathrm{i})(a + b) + (a + b)(a + sb\mathrm{i}) = p$$

so

$$a^2 + s^2b^2 + 2a(a + b) = p \ \Rightarrow 3a^2(3s^2 - 1) = p$$

A1

and

$$(a + sb\mathrm{i})(a - sb\mathrm{i})(a + b) = -q \ \Rightarrow -2a^3(9s^2 + 1) = -q$$

A1

Therefore

$$\frac{p^3}{q^2} = \frac{[3a^2(3s^2-1)]^3}{[2a^3(9s^2+1)]^2} = \frac{27(3s^2-1)^3}{4(9s^2+1)^2}$$

A1* (5)

(iii)

$$y = \frac{(3x-1)^3}{(9x+1)^2}$$

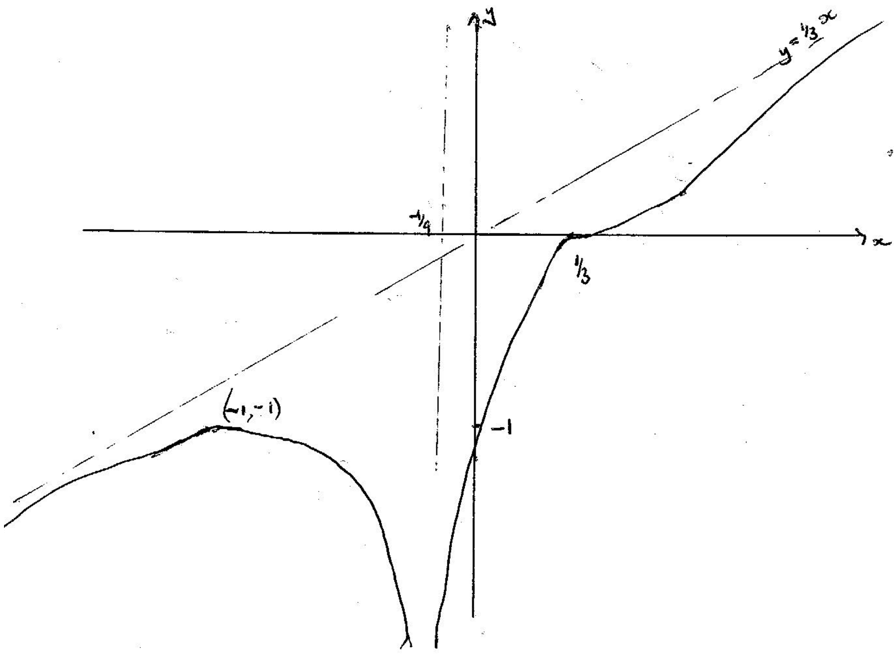
has x intercept at $\left(\frac{1}{3}, 0\right)$, y intercept at $(0, -1)$ **G1** a vertical asymptote at $x = -\frac{1}{9}$ and an asymptote $y = \frac{1}{3}x$ as $x \to \pm\infty$. **G1**

$$\frac{dy}{dx} = \frac{(9x+1)^2 9(3x-1)^2 - (3x-1)^3 18(9x+1)}{(9x+1)^4}$$

$$= \frac{9(3x-1)^2(9x+1-6x+2)}{(9x+1)^3} = \frac{27(3x-1)^2(x+1)}{(9x+1)^3}$$

M1 A1

Thus, the stationary points are a maximum at $(-1,-1)$ and a point of inflection at $\left(\frac{1}{3}, 0\right)$. **G1 G1**



(6)

(iv) If the roots of $z^3 + pz + q = 0$ represent the vertices of an isosceles triangle, then by (ii), $\frac{p^3}{q^2}$ must be real **E1** and as $s^2 > 0$, from (iii) $\frac{p^3}{q^2} > \frac{27}{4} \times -1 = \frac{-27}{4}$ **E1** as $y = \frac{(3x-1)^3}{(9x+1)^2}$ is increasing for $x > 0$. **E1 (3)**

4. (i) By de Moivre,

$$\cos\big((2n+1)\theta\big) + \mathrm{i}\sin\big((2n+1)\theta\big) = (\cos\theta + \mathrm{i}\sin\theta)^{2n+1}$$

Expanding by the binomial theorem and equating real parts

$$\cos\big((2n+1)\theta\big) = \cos^{2n+1}\theta - \binom{2n+1}{2}\cos^{2n-1}\theta\sin^2\theta + \cdots + (-1)^n\binom{2n+1}{2n}\cos\theta\ \sin^{2n}\theta$$

M1 A1

$$= \cos^{2n+1}\theta + \binom{2n+1}{2}\cos^{2n-1}\theta\,(\cos^2\theta - 1) + \cdots + \binom{2n+1}{2n}\cos\theta\ (\cos^2\theta - 1)^n$$

M1

$$= \sum_{r=0}^{n}\binom{2n+1}{2r}\cos^{2n+1-2r}\theta\,(\cos^2\theta - 1)^r$$

A1* (4)

Notice, for (iv), that this expression only contains odd powers of $\cos\theta$.

(ii) The coefficient of x^{2n+1} in $p(x)$ is

$$\sum_{r=0}^{n}\binom{2n+1}{2r}$$

B1

$$(1+x)^{2n+1} = \sum_{r=0}^{2n+1}\binom{2n+1}{r}x^r = \sum_{r=0}^{n}\binom{2n+1}{2r}x^{2r} + \sum_{r=0}^{n}\binom{2n+1}{2r+1}x^{2r+1}$$

Substituting $x = 1$,

$$2^{2n+1} = \sum_{r=0}^{n}\binom{2n+1}{2r} + \sum_{r=0}^{n}\binom{2n+1}{2r+1}$$

and substituting $x = -1$, **M1**

$$0 = \sum_{r=0}^{n}\binom{2n+1}{2r} - \sum_{r=0}^{n}\binom{2n+1}{2r+1}$$

A1

Adding these two results,

$$2^{2n+1} = 2\sum_{r=0}^{n}\binom{2n+1}{2r}$$

and so the required coefficient is 2^{2n} as required. **A1* (4)**

(iii) The coefficient of x^{2n-1} in $\mathrm{p}(x)$ is

$$\sum_{r=0}^{n} -r\binom{2n+1}{2r}$$

B1

As

$$(1+x)^{2n+1} = \sum_{r=0}^{2n+1} \binom{2n+1}{r} x^r = \sum_{r=0}^{n} \binom{2n+1}{2r} x^{2r} + \sum_{r=0}^{n} \binom{2n+1}{2r+1} x^{2r+1}$$

differentiating with respect to x

$$(2n+1)(1+x)^{2n} = \sum_{r=0}^{n} 2r\binom{2n+1}{2r} x^{2r-1} + \sum_{r=0}^{n} (2r+1)\binom{2n+1}{2r+1} x^{2r}$$

M1

Substituting $x = 1$,

$$(2n+1)2^{2n} = \sum_{r=0}^{n} 2r\binom{2n+1}{2r} + \sum_{r=0}^{n} (2r+1)\binom{2n+1}{2r+1}$$

M1

and substituting $x = -1$,

$$0 = -\sum_{r=0}^{n} 2r\binom{2n+1}{2r} + \sum_{r=0}^{n} (2r+1)\binom{2n+1}{2r+1}$$

A1

Subtracting these two results

$$(2n+1)2^{2n} = 2\sum_{r=0}^{n} 2r\binom{2n+1}{2r} = 4\sum_{r=0}^{n} r\binom{2n+1}{2r}$$

and so the required coefficient is

$$-(2n+1)2^{2n} \div 4 = -(2n+1)2^{2n-2}$$

A1* (5)

Alternative

$$\sum_{r=0}^{n} -r\binom{2n+1}{2r} = -\sum_{r=0}^{n} r\,\frac{(2n+1)!}{(2n-2r+1)!\,(2r)!} = -\frac{2n+1}{2}\sum_{r=1}^{n} \frac{(2n)!}{(2n-2r+1)!\,(2r-1)!}$$

M1

$$= -\frac{2n+1}{2}\sum_{r=1}^{n}\binom{2n}{2r-1} = -\frac{2n+1}{2}\sum_{r=0}^{n-1}\binom{2n}{2r+1}$$

M1

As in (ii),

$$\sum_{r=0}^{n-1}\binom{2n}{2r+1} = \frac{1}{2}2^{2n}$$

A1

so

$$\sum_{r=0}^{n} -r\binom{2n+1}{2r} = -\frac{2n+1}{2}\frac{1}{2}2^{2n} = -(2n+1)2^{2n-2}$$

A1* (5)

(iv) Suppose

$$\mathrm{q}(x) = ax^n + bx^{n-1} + cx^{n-2} + \cdots$$

then

$$\mathrm{p}(x) = (x+1)[ax^n + bx^{n-1} + cx^{n-2} + \cdots]^2$$

$$= (x+1)(a^2x^{2n} + 2abx^{2n-1} + (b^2+2ac)x^{2n-2} + \cdots)$$

$$= a^2x^{2n+1} + (a^2+2ab)x^{2n} + (b^2+2ac+2ab)x^{2n-1} + \cdots$$

M1 A1

Thus $a^2 = 2^{2n}$, $a^2 + 2ab = 0$, and $b^2 + 2ac + 2ab = -(2n+1)2^{2n-2}$ **dM1 A1**

Therefore $a = 2^n$ (as $a > 0$), **B1**

$$b = \frac{-a}{2} = -2^{n-1}$$

A1

and

$$2^{2n-2} + 2^{n+1}c - 2^{2n} = -(2n+1)2^{2n-2}$$

so

$$2^{n-3} + c - 2^{n-1} = -(2n+1)2^{n-3}$$

$$c = 2^{n-3}(4 - 1 - 2n - 1) = 2^{n-2}(1-n)$$

as required. **A1*(7)**

5. (i)

$$\frac{1}{x}+\frac{2}{y}=\frac{2}{7}$$

$$7y+14x=2xy$$

$$2xy-7y-14x+49=49$$

$$(2x-7)(y-7)=49$$

B1*

Thus $2x-7=1$, $y-7=49$, or $2x-7=7$, $y-7=7$, or $2x-7=49$, $y-7=1$

M1

and so $(x,y)=(4,56)$, $(7,14)$, or $(28,8)$ **A1 (3)**

(ii)

$$p^2+pq+q^2=n^2$$

$$p^2+2pq+q^2-n^2=pq$$

$$(p+q)^2-n^2=pq$$

$$(p+q+n)(p+q-n)=pq$$

B1*

$p+q+n\neq p$ and $p+q+n\neq q$ as p , q , and n are all positive. $p+q+n>p+q-n$ so $p+q+n\neq 1$ as that would require $p+q-n=pq>1$. **M1**

Thus $p+q+n=pq$ and $p+q-n=1$ as required. **A1***

Therefore $p+q+p+q-1=pq$ **M1**

$$pq-2p-2q+4=3$$

$$(p-2)(q-2)=3$$

dM1

Thus $p-2=1$, $q-2=3$, or $p-2=3$, $q-2=1$

Alternative (I)

$$pq-2p-2q+4=3$$

$$pq-2p-2q+1=0$$

$$p(q-2)=2q-1$$

$$p=\frac{2q-1}{q-2}=2+\frac{3}{q-2}$$

as $q\neq 2$ ($q=2$ would yield -4+4-3=0) so $q-2=1\ or\ 3$ **E1**

and so $(p,q)=(3,5)$, or $(5,3)$ **A1 (6)**

Alternative (II) $p+q+n=pq$ and $p+q-n=1$ yield $p+q=n+1$ and $pq=2n+1$

Therefore, p and q are solutions of $t^2-(n+1)t+(2n+1)=0$

Hence $t=\frac{(n+1)\pm\sqrt{(n+1)^2-4(2n+1)}}{2}=\frac{(n+1)\pm\sqrt{(n-3)^2+12}}{2}$

For integer *t* we require that $(n-3)^2+12$ is a perfect square (in fact an even perfect square).

Thus the difference of squares between $(n-3)^2+12$ and $(n-3)^2$ is 12. Successive squares, z^2 and $(z+1)^2$ differ by $2z+1$, which for $z\geq 6$ is ≥ 13. Thus $(n-3)\leq 5$. Then, either by listing potential solutions exhaustively, or justifying that $(n-3)^2+12$ and $(n-3)^2$ have to be squares differing by $7+5$ and hence $(n-3)^2=2^2$ giving $(p,q)=(3,5)$, or $(5,3)$. **E1 A1 (6)**

(iii) If $p^3+q^3+3pq^2=n^3$, and as p, q, and hence n are all positive, then $p^3<n^3$ and $q^3<n^3$ so $p<n$ and $q<n$, **E1** and hence $p+q-n<p$ and $p+q-n<q$. **A1***

If

$$p^3+q^3+3pq^2+3pq^2=n^3+3pq^2$$

M1

$$(p+q)^3-n^3=3pq^2$$

dM1

$$(p+q-n)((p+q)^2+(p+q)n+n^2)=3pq^2$$

A1

As $p+q-n<p$ and $p+q-n<q$, so $p+q-n=1$ or 3 **A1**

If $p+q-n=1$ then $(n+1)^3-n^3=3pq^2$ and hence $3n^2+3n+1=3pq^2$ **M1** which is not possible as LHS is not a multiple of 3 and RHS is. **E1**

If $p+q-n=3$, then $(n+3)^3-n^3=3pq^2$ and hence $9n^2+27n+27=3pq^2$, that is $3(n^2+3n+3)=pq^2$. **M1** So p or q must divide 3 and hence must be 3 as p and q are prime. **E1**

If $p=3$, then $q-n=0$ but $q<n$ and vice versa if $q=3$ **E1* (11)**

6. (i)

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$\cosh x = \frac{1}{2}(e^x + e^{-x}) = \frac{1}{2}\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots\right) = 1 + \frac{x^2}{2!} + \cdots$$

$$\cosh^2 x = \left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right)^2 = 1 + x^2 + \frac{x^4}{3} + \cdots \geq 1 + x^2$$

B1*

as all terms are of even degree with positive coefficients.

Alternative

$$\cosh x = \frac{1}{2}(e^x + e^{-x}) = 1 + \frac{x^2}{2!} + \cdots \geq 1 + \frac{x^2}{2!}$$

$$\cosh^2 x \geq \left(1 + \frac{x^2}{2!}\right)^2 = 1 + x^2 + \frac{x^4}{4} \geq 1 + x^2$$

B1*

$$\mathrm{f}(x) = \tan^{-1} x - \tanh x$$

$$\mathrm{f}'(x) = \frac{1}{1 + x^2} - \operatorname{sech}^2 x = \frac{\cosh^2 x - (1 + x^2)}{(1 + x^2)\cosh^2 x}$$

M1

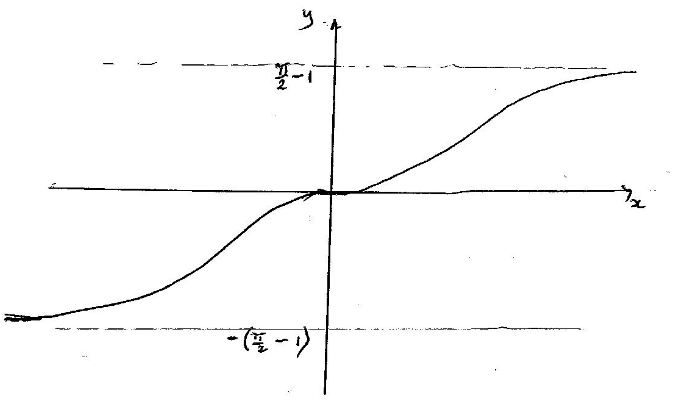
We have shown that the numerator $\cosh^2 x - (1 + x^2) \geq 0$ and the denominator is positive so $\mathrm{f}'(x) \geq 0$ and hence the function f is increasing. **E1***

When $x = 0$, $\mathrm{f}(x) = \mathrm{f}'(x) = 0$ and for all other x, $\mathrm{f}'(x) > 0$

$$\mathrm{f}(-x) = -\mathrm{f}(x)$$

G1

As $x \to \pm\infty$, $\mathrm{f}(x) \to \pm\left(\frac{\pi}{2} - 1\right)$ respectively. **G1 (5)**



(ii) (a)

$$\mathrm{g}(x) = \tan^{-1} x - \frac{1}{2}\pi \tanh x$$

$$\mathrm{g}'(x) = \frac{1}{1+x^2} - \frac{1}{2}\pi \operatorname{sech}^2 x = \frac{2\cosh^2 x - \pi\,(1+x^2)}{2\,(1+x^2)\cosh^2 x}$$

M1

As in (i), the denominator is positive. When $x = 0$, the numerator $= 2 - \pi < 0$. **A1**

The numerator$= (2-\pi)(1+x^2) + 2\left(\frac{x^4}{3} + \cdots\right) \to \infty$ as $x \to \infty$. **M1** Thus, there is a value of $x \neq 0$ for which $\mathrm{g}'(x) = 0$ and as $\mathrm{g}'(x)$ is an even function, there is also the value $-x$. **E1** Hence, there are at least two stationary points for g. **(4)**

Alternative

$\mathrm{g}(0) = 0$ **E1** and $\mathrm{g}(x) \to 0$ as $x \to \infty$ **E1** and $\mathrm{g}(x)$ is not identically zero **E1** so there must be a stationary point for positive x, and similarly for negative. **E1**

(b)

$$\frac{d}{dx}[(1+x^2)\sinh x - x\cosh x] = (1+x^2)\cosh x + 2x\sinh x - x\sinh x - \cosh x$$

M1

$$= x^2\cosh x + x\sinh x \geq 0$$

for $x \geq 0$ **A1**

as $x^2 \geq 0$ and $\cosh x \geq 1$ for all x and $\sinh x \geq 0$ for $x \geq 0$ **E1**

When $x = 0$, $(1+x^2)\sinh x - x\cosh x = 0$ and we have shown $(1+x^2)\sinh x - x\cosh x$ is increasing for $x \geq 0$, thus $(1+x^2)\sinh x - x\cosh x$ is non-negative for $x \geq 0$. **E1 (4)**

(c)

$$\frac{d}{dx}\left[\frac{\cosh^2 x}{1+x^2}\right] = \frac{(1+x^2)2\cosh x\sinh x - 2x\cosh^2 x}{(1+x^2)^2} = \frac{2\cosh x\left((1+x^2)\sinh x - x\cosh x\right)}{(1+x^2)^2}$$

M1 A1

$\frac{2\cosh x}{(1+x^2)^2} > 0$ for all x and by (b) $(1+x^2)\sinh x - x\cosh x$ for $x \geq 0$

so $\frac{\cosh^2 x}{1+x^2}$ is increasing for $x \geq 0$. **E1 (3)**

(d)

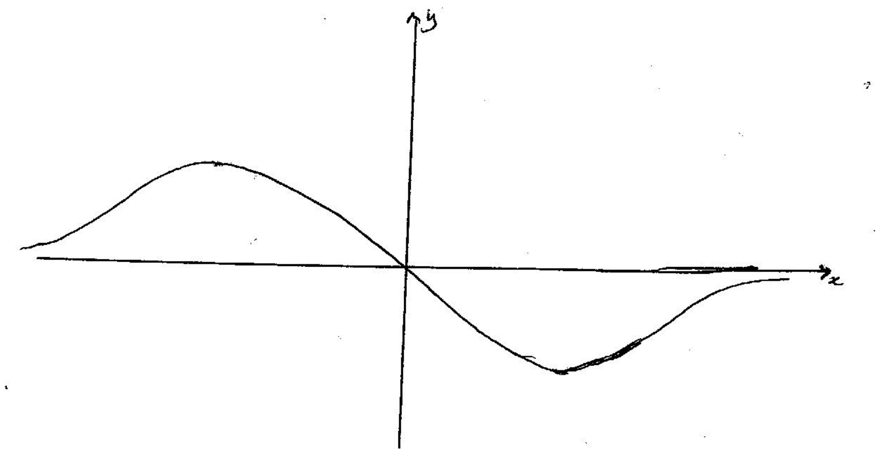
$$\mathrm{g}'(x) = \frac{1}{1+x^2} - \frac{1}{2}\pi \operatorname{sech}^2 x = \frac{1}{\cosh^2 x}\left[\frac{\cosh^2 x}{1+x^2} - \frac{1}{2}\pi\right]$$

By (c) , g' is increasing for $x \geq 0$, and thus there is exactly one value of x for $x > 0$ that $g'(x) = 0$

Similarly, as g' is an even function, there is exactly one value of x for $x < 0$ that $g'(x) = 0$

Thus there are exactly two stationary points. **E1 (1)**

(e)



G3 (3)

7. (i)

Let $x = u^2$, $\frac{dx}{du} = 2u$, $\sqrt{x} = u$ **M1**

$$\int_0^1 \mathrm{f}(\sqrt{x})\, dx = \int_0^1 \mathrm{f}(u)\ 2u\, du = 2\int_0^1 x\mathrm{f}(x)\ dx$$

as required. **A1* (2)**

(ii)

$$\int_0^1 (\mathrm{g}(x) - x)^2\, dx = \int_0^1 \left(\mathrm{g}(x)\right)^2 dx - 2\int_0^1 x\,\mathrm{g}(x)\, dx + \int_0^1 x^2\, dx$$

M1

$$= \int_0^1 \mathrm{g}(\sqrt{x})\, dx - \frac{1}{3} - 2\int_0^1 x\,\mathrm{g}(x)\, dx + \int_0^1 x^2\, dx$$

$$= 2\int_0^1 x\,\mathrm{g}(x)\, dx - \frac{1}{3} - 2\int_0^1 x\,\mathrm{g}(x)\, dx + \left[\frac{x^3}{3}\right]_0^1$$

M1

$$= 0 - \frac{1}{3} + \frac{1}{3} = 0$$

A1*

$(\mathrm{g}(x) - x)^2 \geq 0$

So, the area under the graph of $y = (\mathrm{g}(x) - x)^2 \geq 0$, and the area can only equal zero if $(\mathrm{g}(x) - x)^2 = 0$ for $0 \leq x \leq 1$, that is $\mathrm{g}(x) = x$. **E1 (4)**

(iii)

$$\int_0^1 (\mathrm{h}'(x) - x)^2\, dx = \int_0^1 \left(\mathrm{h}'(x)\right)^2 - 2x\mathrm{h}'(x) + x^2\, dx$$

M1

We are given that

$$\int_0^1 \left(\mathrm{h}'(x)\right)^2 = 2\mathrm{h}(1) - 2\int_0^1 \mathrm{h}(x)\ dx - \frac{1}{3}$$

Integrating by parts

$$\int_0^1 2x\mathrm{h}'(x)\, dx = \ [2x\mathrm{h}(x)]_0^1 - 2\int_0^1 \mathrm{h}(x)\ dx = 2\mathrm{h}(1) - 2\int_0^1 \mathrm{h}(x)\ dx$$

M1 A1

and

$$\int_0^1 x^2\,dx = \left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3}$$

So,

$$\int_0^1 (\mathrm{h}'(x) - x)^2\,dx = 2\mathrm{h}(1) - 2\int_0^1 \mathrm{h}(x)\ dx - \frac{1}{3} - \left(2\mathrm{h}(1) - 2\int_0^1 \mathrm{h}(x)\ dx\right) + \frac{1}{3} = 0$$

A1

As in (ii) with g, $\mathrm{h}'(x) = x$. Thus $\mathrm{h}(x) = \frac{1}{2}x^2 + c$ but $\mathrm{h}(0) = 0$ so $c = 0$ and thus $\mathrm{h}(x) = \frac{1}{2}x^2$

E1 **M1 A1** **A1 (8)**

(iv)

$$\int_0^1 \left(e^{\frac{1}{2}ax}\,k(x) - e^{-\frac{1}{2}ax}\right)^2 dx = \int_0^1 e^{ax}\big(\mathrm{k}(x)\big)^2 - 2\mathrm{k}(x) + e^{-ax}\,dx$$

M1 **dM1**

$$= 2\int_0^1 \mathrm{k}(x)\,dx + \frac{e^{-a}}{a} - \frac{1}{a^2} - \frac{1}{4} - 2\int_0^1 \mathrm{k}(x)\,dx - \left[\frac{e^{-a}}{a}\right]_0^1$$

$$= \frac{e^{-a}}{a} - \frac{1}{a^2} - \frac{1}{4} - \frac{e^{-a}}{a} + \frac{1}{a} = -\frac{1}{a^2} + \frac{1}{a} - \frac{1}{4} = -\frac{4 - 4a + a^2}{4a^2} = -\frac{(2-a)^2}{4a^2}$$

A1 **A1**

As before, $\int_0^1 \left(e^{\frac{1}{2}ax}\,\mathrm{k}(x) - e^{-\frac{1}{2}ax}\right)^2 dx \geq 0$ but $-\frac{(2-a)^2}{4a^2} \leq 0$

Therefore, $\int_0^1 \left(e^{\frac{1}{2}ax}\,\mathrm{k}(x) - e^{-\frac{1}{2}ax}\right)^2 dx = 0$ and $\frac{(2-a)^2}{4a^2} = 0$ **E1**

Thus $e^{\frac{1}{2}ax}\,\mathrm{k}(x) - e^{-\frac{1}{2}ax} = 0$ and $2 - a = 0$

So $a = 2$ and $\mathrm{k}(x) = e^{-ax} = e^{-2x}$ **A1 (6)**

8. (i)

$$y = xe^{-x}$$

$$\frac{dy}{dx} = e^{-x} - xe^{-x} = (1-x)e^{-x}$$

$$\frac{d^2y}{dx^2} = -e^{-x} - (1-x)e^{-x} = (x-2)e^{-x}$$

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = (x-2)e^{-x} + 2(1-x)e^{-x} + xe^{-x} = 0$$

M1 A1

$$x = 0, \qquad y = xe^{-x} = 0, \qquad \frac{dy}{dx} = (1-x)e^{-x} = 1$$

B1*

For $x \le 1$, $(1-x) \ge 0$, $e^{-x} > 0$, so $\frac{dy}{dx} = (1-x)e^{-x} \ge 0$ **E1 (4)**

(ii) From (i),

$$\mathrm{g}_1(x) = xe^{-x}$$

B1

Consider

$$y = \mathrm{g}_2(x) = (a+bx)e^{x}$$

for $x \ge 1$ **B1**

Then g_2 must be a solution of $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 0$, $\mathrm{g}_1(1) = \mathrm{g}_2(1)$, and $\mathrm{g'}_1(1) = \mathrm{g'}_2(1)$

$$\frac{dy}{dx} = be^{x} + (a+bx)e^{x} = \big((a+b)+bx\big)e^{x}$$

$$\frac{d^2y}{dx^2} = be^{x} + \big((a+b)+bx\big)e^{x} = \big((a+2b)+bx\big)e^{x}$$

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = \big((a+2b)+bx\big)e^{x} - 2\big((a+b)+bx\big)e^{x} + (a+bx)e^{x} = 0$$

as required.

$$\mathrm{g}_1(1) = \mathrm{g}_2(1) \Rightarrow e^{-1} = (a+b)e$$

$$\mathrm{g'}_1(1) = \mathrm{g'}_2(1) \Rightarrow 0 = (a+2b)e$$

M1 A1

So $a = -2b$ and thus $b = -e^{-2}$

Hence, $\mathrm{g}_2(x) = (2e^{-2} - e^{-2}x)e^{x} = (2-x)e^{x-2}$ **A1ft (5)**

(iii) $y = \mathrm{g}_2(x)$ is a reflection of $y = \mathrm{g}_1(x)$ in $x = 1$, **B1** which can be justified by substituting for x using $x' = 2 - x$ in $y = \mathrm{g}_1(x)$, $xe^{-x} = (2-x')e^{x'-2}$ as expected. **E1 (2)**

(iv) If $y = \mathrm{k}(c-x)$, then $\frac{dy}{dx} = -\mathrm{k}'(c-x)$, and $\frac{d^2y}{dx^2} = \mathrm{k}''(c-x)$ **M1**

So $\frac{d^2y}{dx^2} - p\frac{dy}{dx} + qy = \mathrm{k}''(c-x) + p\mathrm{k}'(c-x) + q\mathrm{k}(c-x) = 0$ **A1** provided that $r \le c - x \le s$

i.e. $c - s \le x \le c - r$ **B1 (3)**

(v) If $\mathrm{h}(x) = e^{-x}\sin x$, then $\mathrm{h}'(x) = -e^{-x}\sin x + e^{-x}\cos x$,

so $\mathrm{h}'\left(\frac{\pi}{4}\right) = -e^{-\frac{\pi}{4}}\sin\frac{\pi}{4} + e^{-\frac{\pi}{4}}\cos\frac{\pi}{4} = -\frac{1}{\sqrt{2}}e^{-\frac{\pi}{4}} + \frac{1}{\sqrt{2}}e^{-\frac{\pi}{4}} = 0$ as required. **B1***

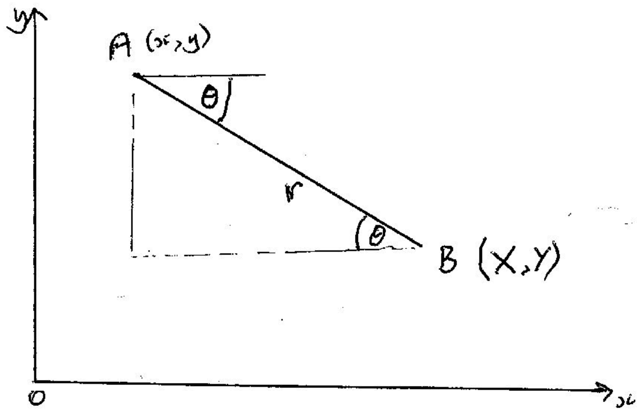
(a) Using (iv), the solution for $\frac{1}{4}\pi \le x \le \frac{5}{4}\pi$ must be $y = e^{-(c-x)}\sin(c-x)$ **M1** where

$-\frac{3}{4}\pi \le (c-x) \le \frac{1}{4}\pi$. **M1** That is $c = \frac{1}{2}\pi$. So $y = e^{x-\frac{1}{2}\pi}\cos x$ **A1**

(b) Similarly, the solution for $\frac{5}{4}\pi \le x \le \frac{9}{4}\pi$ must be $y = e^{c-x-\frac{1}{2}\pi}\cos(c-x)$ where

$\frac{1}{4}\pi \le (c-x) \le \frac{5}{4}\pi$. That is $c = \frac{5}{2}\pi$. **B1** So $y = e^{2\pi - x}\sin x$ **B1 (6)**

9. (i)



G1 (1)

(ii)

$$X = x + r\cos\theta$$

$$\dot{X} = \dot{x} - r\sin\theta\,\dot{\theta}$$

$$\ddot{X} = \ddot{x} - r\cos\theta\ \dot{\theta}^2 - r\sin\theta\ \ddot{\theta}$$

$$Y = y - r\sin\theta$$

$$\dot{Y} = \dot{y} - r\cos\theta\ \dot{\theta}$$

$$\ddot{Y} = \ddot{y} + r\sin\theta\ \dot{\theta}^2 - r\cos\theta\ \ddot{\theta}$$

B1 B1 (2)

(iii) The acceleration of A perpendicular to the string is $\ddot{x}\ \sin\theta + \ddot{y}\ \cos\theta$, **E1** and likewise for B is $\ddot{X}\ \sin\theta + \ddot{Y}\ \cos\theta$ so resolving for each in that direction, $\ddot{x}\ \sin\theta + \ddot{y}\ \cos\theta = 0$ and $\ddot{X}\ \sin\theta + \ddot{Y}\ \cos\theta = 0$ as only force is parallel to the string. **E1**

(alternatively, resolving forces in x,y for both particles, and adding necessary equations gives both results – need to show each equation for each E mark)

Substituting for $\ddot{X}$ and $\ddot{Y}$ using the results in (i),

$$(\ddot{x} - r\cos\theta\ \dot{\theta}^2 - r\sin\theta\ \ddot{\theta})\sin\theta + (\ddot{y} + r\sin\theta\ \dot{\theta}^2 - r\cos\theta\ \ddot{\theta}\,)\cos\theta = 0$$

M1

(also may just notice $\sin\theta\ (\ddot{X} - \ddot{x}) + \cos\theta\ (\ddot{Y} - \ddot{y}) = 0$)

Thus

$$(\ddot{x}\ \sin\theta + \ddot{y}\ \cos\theta) - r\ddot{\theta} = 0$$

A1ft

and so $r\ddot{\theta} = 0$, i.e. $\ddot{\theta} = 0$ **A1**

Integrating, $\dot{\theta} = k$ Initially, $\dot{y} = u$ and $\dot{Y} = 0$ when $\theta = 0$ so using $\dot{Y} = \dot{y} - r\cos\theta\ \dot{\theta}$, initially $0 = u - r\ \dot{\theta}$ and so $k = \frac{u}{r}$. **M1 A1**

$$\dot{\theta} = \frac{u}{r}$$

and so, integrating, $\theta = \frac{u}{r}\,t + c$, and using the initial conditions, $c = 0$

Hence,

$$\theta = \frac{ut}{r}$$

as required. **M1 A1* (9)**

(iv) Resolving in the x direction for m, $m\ddot{x} = T\cos\theta$, and for M, $M\ddot{X} = -T\cos\theta$, so adding, $m\ddot{x} + M\ddot{X} = 0$. Likewise in the y direction, $m\ddot{y} = -T\sin\theta$, $M\ddot{Y} = T\sin\theta$, giving $m\ddot{y} + M\ddot{Y} = 0$. **E1**

Integrating this, $m\dot{y} + M\dot{Y} = mu$, using initial conditions. Integrating again and applying initial conditions, $my + MY = mut$. **M1 A1 (3)**

(v) As $Y = y - r\sin\theta$, $my + M(y - r\sin\theta) = mut$, so $my + My - Mr\sin\left(\frac{ut}{r}\right) = mut$ and thus,

$$y = \frac{1}{m+M}\left(mut + Mr\sin\left(\frac{ut}{r}\right)\right)$$

E1 A1* (2)

(vi) Differentiating,

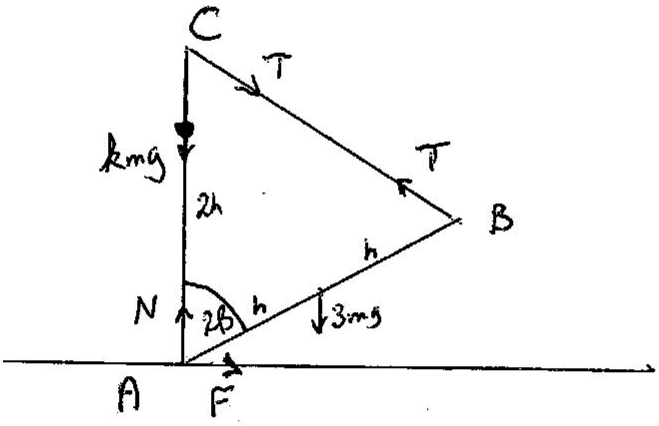
$$\dot{y} = \frac{1}{m+M}\left(mu + Mr\,\frac{u}{r}\cos\left(\frac{ut}{r}\right)\right) = \frac{u}{m+M}\left(m + M\cos\left(\frac{ut}{r}\right)\right)$$

M1 A1

When $\left(\frac{ut}{r}\right) = \pi$, $\dot{y} = \frac{u}{m+M}\,(m - M) < 0$ if $M > m$ as required.

E1 (3)

10.



G3

(i) Resolving vertically for the particle $T = kmg$ **B1**

Taking moments about A for the beam $3mgh\sin 2\beta = T2h\cos\beta$ **M1A1**

Thus $k = 3\sin\beta$ **A1 (7)**

Resolving horizontally for the beam $F = T\cos\beta = kmg\cos\beta$ **M1**

Resolving vertically for the beam $N + T\sin\beta = 3mg$ **B1**

Thus $N = 3mg - kmg\sin\beta = 3mg - 3mg\sin^2\beta = 3mg\cos^2\beta$ **A1**

As $F \le \mu N$, $kmg\cos\beta \le \mu 3mg\cos^2\beta$ so $k \le 3\mu\cos\beta$ **M1**

Thus $k^2 \le 9\mu^2\cos^2\beta = \mu^2(9 - 9\sin^2\beta) = \mu^2(9 - k^2) = 9\mu^2 - \mu^2k^2$

So $k^2 + \mu^2k^2 \le 9\mu^2$ and so $k^2 \le \frac{9\mu^2}{\mu^2+1}$ as required. **M1 A1* (6)**

Alternative

Considering, total force at A as R, there are three forces acting on the beam which must be concurrent, **M1** and so line of action of R passes through midpoint of BC, **A1** and thus the angle of friction must be at least β . **A1** $\mu \ge \tan\beta = \frac{k/3}{\sqrt{1-(k/3)^2}}$ **M1** so $\mu^2 \ge \frac{k^2}{9-k^2} \Rightarrow k^2 \le \frac{9\mu^2}{\mu^2+1}$ **M1A1 (6)**

(ii) From (i) $F = T\cos\beta = kmg\cos\beta = 2mg\cos\beta$ **B1**

Moments about A for the beam $3mgh\sin 2\beta + mgxh\sin 2\beta = T2h\cos\beta = 4mgh\cos\beta$

Hence $3\sin\beta + x\sin\beta = 2$ and thus $\sin\beta = \frac{2}{3+x}$ **M1A1**

Resolving vertically $N + T\sin\beta = 3mg + mg$ and so $N = 4mg - 2mg\frac{2}{3+x} = 4mg\frac{2+x}{3+x}$

$$\frac{F^2}{N^2} = \frac{4m^2g^2\cos^2\beta}{16m^2g^2}\frac{(3+x)^2}{(2+x)^2} = \frac{(3+x)^2}{4(2+x)^2}\left(1 - \left(\frac{2}{3+x}\right)^2\right)$$

$$= \frac{1}{4(2+x)^2}\left((3+x)^2 - 4\right) = \frac{x^2+6x+5}{4(2+x)^2}$$

as required. **M1 A1***

$$\frac{1}{3} - \frac{F^2}{N^2} = \frac{4(2+x)^2 - 3(x^2+6x+5)}{12(2+x)^2} = \frac{x^2-2x+1}{12(2+x)^2} = \frac{(x-1)^2}{12(2+x)^2} \geq 0$$

Thus

$$\frac{F^2}{N^2} \leq \frac{1}{3}$$

and so to be in equilibrium whatever the value of x , we require $\mu \geq \frac{1}{\sqrt{3}}$ and hence $\frac{1}{\sqrt{3}}$ is the minimum value of μ . **M1A1(7)**

11.

$$\sum_{k=1}^{\infty} \frac{k+1}{k!} x^k = \sum_{k=1}^{\infty} \frac{x^k}{k!} + \sum_{k=1}^{\infty} \frac{x^k}{(k-1)!}$$

M1

$$= \sum_{k=0}^{\infty} \frac{x^k}{k!} - 1 + x \sum_{k=1}^{\infty} \frac{x^{k-1}}{(k-1)!}$$

M1

$$= e^x - 1 + x \sum_{k=0}^{\infty} \frac{x^k}{k!} = e^x - 1 + xe^x = (x+1)e^x - 1$$

as required. **A1* (3)**

Alternative

$$\sum_{k=1}^{\infty} \frac{k+1}{k!} x^k = \frac{d}{dx}\left(\sum_{k=1}^{\infty} \frac{x^{k+1}}{k!}\right)$$

M1

$$= \frac{d}{dx}\left(x \sum_{k=0}^{\infty} \frac{x^k}{k!} - 1\right) = \frac{d}{dx}\left(x(e^x - 1)\right)$$

M1

$$= (x+1)e^x - 1$$

as required. **A1* (3)**

(i) (a) $\mathrm{P}(D=0) = \mathrm{P}(N=0) = e^{-n}$ **B1**

(b)

$$\mathrm{E}(D) = \sum_{d=1}^{\infty} d \; \mathrm{P}(D=d)$$

M1

$$\mathrm{P}(D=d) = \sum_{k=d}^{\infty} \mathrm{P}(D=d|Y=k) \; \mathrm{P}(Y=k) = \sum_{k=d}^{\infty} \frac{1}{k} \frac{n^k e^{-n}}{k!}$$

M1

So

$$\mathrm{E}(D) = \sum_{d=1}^{\infty} d \sum_{k=d}^{\infty} \frac{1}{k} \frac{n^k e^{-n}}{k!}$$

as required. **A1*(4)**

$$\sum_{d=1}^{\infty} \sum_{k=d}^{\infty} \quad = (1,1) + (1,2) + \cdots + (2,2) + (2,3) + \cdots + (3,3) + (3,4) + \cdots$$

$$= (1,1) + (1,2) + (2,2) + (1,3) + (2,3) + (3,3) + \cdots$$

$$=\sum_{k=1}^{\infty}\sum_{d=1}^{k}$$

E1 A1

So

$$\mathrm{E}(D)=\sum_{d=1}^{\infty} d\sum_{k=d}^{\infty}\frac{1}{k}\frac{n^k e^{-n}}{k!}=\sum_{k=1}^{\infty}\sum_{d=1}^{k} d\,\frac{1}{k}\frac{n^k e^{-n}}{k!}=\sum_{k=1}^{\infty}\frac{1}{k}\frac{n^k e^{-n}}{k!}\sum_{d=1}^{k} d$$

A1*(3)

(c)

Thus

$$\mathrm{E}(D)=\sum_{k=1}^{\infty}\frac{1}{k}\frac{n^k e^{-n}}{k!}\frac{k(k+1)}{2}=\frac{e^{-n}}{2}\sum_{k=1}^{\infty}\frac{k+1}{k!}n^k=\frac{e^{-n}}{2}\left((n+1)e^n-1\right)$$

B1 **M1 A1**

by using the result of the stem

$$=\frac{1}{2}\left(n+1-e^{-n}\right)$$

A1*(4)

(ii) (a)

$$\mathrm{P}(Z=0)=\sum_{k=1}^{n}\mathrm{P}(Z=0|X_n=X_k)\ \mathrm{P}(X_n=X_k)=\sum_{k=1}^{n}\frac{1}{n}e^{-k}$$

M1

$$=\frac{1}{n}e^{-1}\frac{1-e^{-n}}{1-e^{-1}}=\frac{1-e^{-n}}{n(e-1)}$$

A1 (2)

(b)

$$\mathrm{E}(Z)=\sum_{s=1}^{\infty} s\,\mathrm{P}(Z=s)=\sum_{s=1}^{\infty} s\sum_{k=1}^{n}\frac{1}{n}\mathrm{P}(X_k=s)$$

M1

$$=\frac{1}{n}\sum_{k=1}^{n}\sum_{s=1}^{\infty} s\mathrm{P}(X_k=s)$$

M1

$$=\frac{1}{n}\sum_{k=1}^{n} k\ =\frac{1}{n}\frac{n(n+1)}{2}=\frac{1}{2}(n+1)>\frac{1}{2}\left(n+1-e^{-n}\right)=\mathrm{E}(D)$$

A1 **A1(4)**

12. (i) There are $\binom{2n}{2k}$ ways of choosing $2k$ socks from $2n$. **E1** If there is no pair of socks, then the $2k$ socks must be of different colours; the colours can be chosen $\binom{n}{2k}$ ways and for each colour there are 2 ways of choosing a sock of that colour. **E1** Hence the probability of no pairs is

$$\frac{\binom{n}{2k}2^{2k}}{\binom{2n}{2k}}$$

B1 (3)

Alternative

The probability that all socks chosen do not include any pairs is

$$\frac{2n}{2n}\times\frac{2n-2}{2n-1}\times\frac{2n-4}{2n-2}\times\cdots\times\frac{2n-2(2k-1)}{2n-2k+1}$$

as having removed r different socks leaves only $2n-2r$ possibilities from the remaining $2n-r$.

E1

$$\frac{2n}{2n}\times\frac{2n-2}{2n-1}\times\frac{2n-4}{2n-2}\times\cdots\times\frac{2n-2(2k-1)}{2n-2k+1}=\frac{2^{2k}n(n-1)\cdots(n-2k+1)}{(2n)!\big/(2n-2k)!}$$

$$=\frac{2^{2k}n!}{(n-2k)!}\div\frac{(2n)!}{(2n-2k)!}=2^{2k}\times\frac{n!}{(n-2k)!\,(2k)!}\div\frac{(2n)!}{(2n-2k)!\,(2k)!}$$

E1

$$=\frac{\binom{n}{2k}2^{2k}}{\binom{2n}{2k}}$$

B1 (3)

(ii) For $X_{n,k}=r$, there must be $2r$ socks that are pairs and $2k-2r$ that are of different colours. **E1**

The colours of the pairs can be chosen $\binom{n}{r}$ ways and the colours of the remaining $2k-2r$ individual socks can be chosen from the remaining $n-r$ colours $\binom{n-r}{2k-2r}$ ways **E1**: for each colour chosen for an individual sock there are two choices of which sock of the pair is chosen. **E1**

Hence,

$$\mathrm{P}(X_{n,k}=r)=\frac{\binom{n}{r}\binom{n-r}{2k-2r}2^{2k-2r}}{\binom{2n}{2k}}=\frac{\binom{n}{r}\binom{n-r}{2(k-r)}2^{2(k-r)}}{\binom{2n}{2k}}$$

A1*(4)

(iii)

$$\frac{k(2k-1)}{2n-1}\mathrm{P}\left(X_{n-1,k-1}=r-1\right)=\frac{k(2k-1)}{2n-1}\frac{\binom{n-1}{r-1}\binom{n-r}{2(k-r)}2^{2(k-r)}}{\binom{2n-2}{2k-2}}$$

M1A1

$$=\frac{k(2k-1)(n-1)!\,(2k-2)!\,(2n-2k)!}{(2n-1)(n-r)!\,(r-1)!\,\,(2n-2)!}\binom{n-r}{2(k-r)}2^{2(k-r)}$$

A1

$$=\frac{1}{(2n-1)(2n-2)!}\frac{k(2k-1)(2k-2)!}{(r-1)!}\frac{(n-1)!\,(2n-2k)!}{(n-r)!}\binom{n-r}{2(k-r)}2^{2(k-r)}$$

M1A1

$$=\frac{1}{(2n-1)!}\frac{k(2k-1)!}{(r-1)!}\frac{(n-1)!\,(2n-2k)!}{(n-r)!}\binom{n-r}{2(k-r)}2^{2(k-r)}$$

M1

$$=\frac{2n}{(2n)!}\frac{(2k)!}{2}\frac{(n-1)!\,(2n-2k)!}{(r-1)!\,(n-r)!}\binom{n-r}{2(k-r)}2^{2(k-r)}$$

$$=\frac{2n(n-1)!}{2(2n)!}\frac{r(2n-2k)!}{r!\,(n-r)!}\frac{(2k)!}{1}\binom{n-r}{2(k-r)}2^{2(k-r)}$$

$$=r\frac{n!}{r!\,(n-r)!}\frac{(2n-2k)!\,(2k)!}{(2n)!}\binom{n-r}{2(k-r)}2^{2(k-r)}$$

A1

$$=r\frac{\binom{n}{r}\binom{n-r}{2(k-r)}2^{2(k-r)}}{\binom{2n}{2k}}=r\mathrm{P}\left(X_{n,k}=r\right)$$

A1*(8)

Or alternatively, in the OPPOSITE DIRECTION

$$r\mathrm{P}\left(X_{n,k}=r\right)=r\frac{\binom{n}{r}\binom{n-r}{2(k-r)}2^{2(k-r)}}{\binom{2n}{2k}}$$

M1A1

$$=r\frac{n!}{r!\,(n-r)!}\frac{(2n-2k)!\,(2k)!}{(2n)!}\binom{n-r}{2(k-r)}2^{2(k-r)}$$

A1

$$=\frac{2n(n-1)!}{2(2n)!}\frac{r(2n-2k)!}{r!\,(n-r)!}\frac{(2k)!}{1}\binom{n-r}{2(k-r)}2^{2(k-r)}$$

M1A1

$$= \frac{2n}{(2n)!}\frac{(2k)!}{2}\frac{(n-1)!\,(2n-2k)!}{(r-1)!\,(n-r)!}\binom{n-r}{2(k-r)}2^{2(k-r)}$$

M1

$$= \frac{1}{(2n-1)!}\frac{k(2k-1)!}{(r-1)!}\frac{(n-1)!\,(2n-2k)!}{(n-r)!}\binom{n-r}{2(k-r)}2^{2(k-r)}$$

$$= \frac{1}{(2n-1)(2n-2)!}\frac{k(2k-1)(2k-2)!}{(r-1)!}\frac{(n-1)!\,(2n-2k)!}{(n-r)!}\binom{n-r}{2(k-r)}2^{2(k-r)}$$

$$= \frac{k(2k-1)(n-1)!\,(2k-2)!\,(2n-2k)!}{(2n-1)(n-r)!\,(r-1)!\,\,(2n-2)!}\binom{n-r}{2(k-r)}2^{2(k-r)}$$

$$= \frac{k(2k-1)}{2n-1}\frac{\binom{n-1}{r-1}\binom{n-r}{2(k-r)}2^{2(k-r)}}{\binom{2n-2}{2k-2}}$$

A1

$$= \frac{k(2k-1)}{2n-1}\mathrm{P}(X_{n-1,k-1} = r-1)$$

A1*(8)

$$\mathrm{E}(X_{n,k}) = \sum_{r=0}^{k} r\mathrm{P}(X_{n,k} = r) = \sum_{r=1}^{k} r\mathrm{P}(X_{n,k} = r) = \sum_{r=1}^{k}\frac{k(2k-1)}{2n-1}\mathrm{P}(X_{n-1,k-1} = r-1)$$

M1 **M1A1**

$$= \frac{k(2k-1)}{2n-1}\sum_{r=0}^{k-1}\mathrm{P}(X_{n-1,k-1} = r) = \frac{k(2k-1)}{2n-1}$$

M1 **A1(5)**